Frisch Scheme Identification of Robots Dynamic Parameters

Stefano Massaroli[1], Federico Califano[2],
Claudio Melchiorri[3], Atsushi Yamashita[1], and Hajime Asama[1]

[1] Department of Precision Engineering, The University of Tokyo,
7-3-1 Hongo, Bunkyo, Tokyo 113-8656, Japan.
{massaroli, yamashita, asama}@robot.t.u-tokyo.ac.jp
[2] Faculty of Electrical Engineering, Mathematics & Computer Science (EWI),
Robotics and Mechatronics (RAM), University of Twente
Hallenweg 23, Enschede 7522NH, The Netherlands
f.califano@utwente.nl
[3] Department of Electrical, Electronic and Information Engineering (DEI),
University of Bologna
Viale Risorgimento 2, Bologna 40136, Italy
claudio.melchiorri@unibo.it

Abstract. In this paper a novel procedure for the identification of the dynamic parameters of rigid robot manipulators is presented. The method is developed according to the *Frisch scheme*, an estimation procedure based on *a priori* assumptions on the data that are different with respect to the usual ones, e.g. to the common *Least Square* (LS) method. The peculiarities of this scheme give rise to a new way to perform identification on a robot manipulator and offer a better insight of its results with respect to classical methods. A complete and systematic identification procedure is presented involving and improving known techniques of both model reduction and planning of optimal exciting trajectory. Simulation results performed on a 2-DOF robotic manipulator are presented and discussed, confirming the effectiveness of the proposed method.

Keywords: Robotics, System Identification, Frisch Scheme, Optimal Trajectory Planning

1 Introduction

Advanced control techniques, simulations and diagnostics of robots often require an accurate knowledge of the dynamic model. Although robot manipulators are designed according to precise mechanical specifications, the dynamic parameters of their links are typically not perfectly known, even by robot manufacturers. In order to achieve a correct knowledge of those parameters, experimental identification techniques can be performed where the estimates are obtained from direct measurements collected during the motion of the robot. The topic has been widely developed in the literature (see e.g [1,2,3,4]) and the great amount of different contributions is due to the variety of steps and degrees of freedom

that are present in identification procedures. The motivation of this work is driven by the consideration that apparently most of the effort was done to improve technical aspects preceding the real identification step, i.e. reduction of the model and generation of exciting trajectories. Indeed the identification of the parameters has been basically performed by means of *least squares* (LS) estimation techniques. However, these methods are not always reliable and often lead to physically infeasible estimates [5].

The aim of this work is to improve the identification step, using tools from system identification theory not exploited yet in robotics, in particular the (algebraic) *Frisch Scheme* [6]. This estimation scheme falls into the Errors-in-Variable (EIV) context and abstracts from a commonly chosen input/output formulation of the dynamic model of robot manipulators. Indeed, it is based on milder and more general assumptions than those implicitly done when LS based identification procedures are performed. This method introduces some *post-identification* degree of freedom in the selection of dynamic parameters of the manipulator among the feasible solutions compatible with the scheme itself. This aspect represents a resource since physical knowledge of the dynamic model of the robot can help in this selection, mixing a pure *black box* estimation step with a reasonable *white box* selection procedure.

The main contribution of this paper lies in the introduction of the Frisch Scheme in a robot identification framework. Moreover, a systematic procedure that involves and improves known techniques of model reduction and exciting trajectory generation is presented. The paper is organized as follows: Sec. 2 summarizes some results of EIV schemes, in particular of the Frisch scheme. In Sec. 3 the proposed identification procedure on robot manipulators is presented: in Sec. 3.1 and Sec. 3.2 technical issues about model reduction and planning of exciting trajectory are discussed and in Sec. 3.3 the real identification step is presented. Sec. 4 reports a simulation example validating the proposed method on 2-DOF robotic arm. Finally, conclusions and future work are reported in Sec. 5.

Notation: The set $\mathbb{R}$ is the the set of real, $\|\cdot\|_2$ is the norm induced by the inner product of $\mathbb{R}^n$ while $\|\cdot\|_\infty$ is the infinity matrix norm. The origin of $\mathbb{R}^n$ is $\mathbb{0}_n$ while the n-dimensional identity matrix is $\mathbf{I}_n$. $\mathcal{N}(\mathbb{0}_n, \mathbf{I}_n)$ is a multivariate normal distribution with zero mean and unitary variance. Consider a square matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ and a vector $\mathbf{v} \in \mathbb{R}^n$; $\mathbf{diag}(\mathbf{M}) \in \mathbb{R}^n$ denotes the diagonal of $\mathbf{M}$ while $\mathbf{diag}(\mathbf{v}) \in \mathbb{R}^{n\times n}$ is the diagonal matrix whose diagonal is $\mathbf{v}$. Moreover, $\mathbf{ker}(\mathbf{M})$ denotes a basis of the kernel (null space) of $\mathbf{M}$; therefore, $\mathbf{ker}(\cdot)$ represents the operator of kernel basis extraction. Scalars are denoted as lower–case letters, vectors as bold and lower–case and matrices as bold capital letters.

2 The Frisch Estimation Scheme: Background

Most of the content of this section derives from [6]. The Frisch Scheme is an estimation scheme able to extract linear relations from data affected by noise.

As every systematic procedure that deduces a model starting from data affected by errors, it relies on the *modification* of the observed data since a deterministic mathematical relation able to fit all the observations does not exist. A major difference among different estimation schemes is the set of *a priori* assumptions that are necessary to estimate linear relations from noisy data. Different schemes modify observations in different ways. In [7] a detailed analysis of the assumptions behind several schemes is present. The Frisch Scheme falls into the Errors-in-Variables (EIV) context that assumes additive noise on *all* variables.

2.1 Assumptions for EIV Estimation Schemes

Consider the following linear algebraic equation:

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0 \tag{1}$$

where n variables x_i are linked to n scalars α_i. Let us collect m measurements of the variables. We can define the *observation matrix* as

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}. \tag{2}$$

From equation (1), it holds

$$\mathbf{X}\mathbf{A} = \mathbb{0}_n \tag{3}$$

where $\mathbf{A} = [\alpha_1,\ \alpha_2, \cdots,\ \alpha_n]^\top$ [4]. By defining the *sample covariance matrix* $\mathbf{\Sigma} \triangleq \frac{1}{m}\mathbf{X}^\top\mathbf{X}$, it can be noticed that (3) is equivalent to

$$\mathbf{\Sigma}\mathbf{A} = \mathbb{0}_n \tag{4}$$

as $\mathbf{\Sigma}$ and $\mathbf{X}$ share the same kernel. The diagonal of $\mathbf{\Sigma}$ contains the variances of the variables while the covariances are the off-diagonal elements. Assume now that the noise enters in an additive way in the variables, i.e. $x_i = \hat{x}_i + \tilde{x}_i$ where $\hat{x}_i$ is the unknown exact value of the variable and $\tilde{x}_i$ is the corresponding noise sample. Notice that, due to the noise, $\mathbf{\Sigma} \succ 0$ and it is thus impossible to extract any linear relation from its kernel linear relations. To do that, it is necessary to modify the observations covariance $\mathbf{\Sigma}$. Under the usual assumptions made in EIV schemes of zero–mean noise and statistical independence between noise and noisless samples (see [6]) it turns out that $\mathbf{\Sigma} = \hat{\mathbf{\Sigma}} + \tilde{\mathbf{\Sigma}}$, with $\hat{\mathbf{\Sigma}} \triangleq \frac{1}{m}\hat{\mathbf{X}}^\top\hat{\mathbf{X}}$ and $\tilde{\mathbf{\Sigma}} \triangleq \frac{1}{m}\tilde{\mathbf{X}}^\top\tilde{\mathbf{X}}$, where $\hat{\mathbf{X}}$, $\tilde{\mathbf{X}}$ are defined as in (2) with $\hat{x}_i$ and $\tilde{x}_i$ instead of x_i. The problem of finding linear relations compatible with noisy data, namely the "estimation problem", formulated by Rudolf E. Kálmán, is the following:

[4] Note that, in general, $\mathbf{A} \in \mathbb{R}^{n \times p}$ whenever there exists p independent linear relations compatible with the data.

Definition 1 (Estimation Problem [7]). *Given* $\mathbf{\Sigma}$*, determine* $\tilde{\mathbf{\Sigma}}$ *such that*

$$\hat{\mathbf{\Sigma}} = \mathbf{\Sigma} - \tilde{\mathbf{\Sigma}} \geq 0 \qquad \det(\hat{\mathbf{\Sigma}}) = 0 \tag{5}$$

Any base of $\ker(\hat{\mathbf{\Sigma}})$ *will span a space describing a set of linear relations compatible with the data and the assumptions, i.e.* $\hat{\mathbf{\Sigma}}\mathbf{A} = \mathbb{0}_n$ *can now be solved.*

Differences among estimation schemes lie in the assumptions made on the noise variables and consequently on $\tilde{\mathbf{\Sigma}}$. Two estimation schemes of interest in this work are now described.

2.2 Ordinary Least Square (OLS) [8]

The assumption made in this estimation scheme is that only one variable is affected by noise, while the remaining ones are noise-free, i.e.

$$\tilde{\mathbf{\Sigma}} = \mathrm{diag}[0, \cdots, 0, \tilde{\sigma}_i^2, 0, \cdots, 0]$$

where $\tilde{\sigma}_i^2$ is the variance of $\tilde{x}_i$. The solution of the estimation problem is given by

$$\tilde{\sigma}_i^2 = \frac{\det(\mathbf{\Sigma})}{\det(\mathbf{\Sigma}_i)} \tag{6}$$

where $\mathbf{\Sigma}_i$ is the matrix built by deleting the i-th row and the i-th column of $\mathbf{\Sigma}$. In fact $\tilde{\sigma}_i^2$, as defined in (6), is the maximum amount of noise compatible with condition (5). Note that:

- When no assumption is made about the noisy variable, the ordinary least squares scheme leads to n different solutions;
- Each OLS solution $\mathbf{A}_i$ $(i = 1, \ldots, n)$ minimizes the squared estimation error, i.e.

$$\mathbf{A}_i = \underset{\mathbf{A}_i}{\arg\min} \| \mathbf{X}_i - \tilde{\mathbf{X}}_i \mathbf{A}_i \|_2^2 \tag{7}$$

  where $\mathbf{X}_i$ is the i-th column of $\mathbf{X}$ and $\tilde{\mathbf{X}}_i$ is obtained by deleting the i-th column of $\mathbf{X}$.

2.3 The Frisch Scheme

This scheme relaxes the OLS assumptions on the noise by requiring only the mutual independence of the noise variables affecting dirrent variables, i.e.

$$\tilde{\mathbf{\Sigma}} = \mathrm{diag}\left[\tilde{\sigma}_1^2, \tilde{\sigma}_2^2, \cdots, \tilde{\sigma}_n^2\right]$$

Notice that, for a given sample covariance matrix, $\mathbf{\Sigma}$ the n OLS solutions are included in the solutions of the Frisch estimation scheme. This means that the *a priori* assumptions made for the Frisch scheme are milder and more general than those of any estimator which provides a single solution like *least squares* (either computed by pseudoinversion or recursively). On the other hand, in this

way the computation of closed-form solutions is more difficult, since in general there are infinite possible solutions. In order to overcome this problem, we use some of the results reported in [6], where this approach is presented in details. In particular, we need to introduce a definition and a theorem. Let's define $\text{Maxcor}_{\mathcal{F}}(\boldsymbol{\Sigma})$ as the maximum number of linear relations that can be extracted from $\boldsymbol{\Sigma}$ under the assumptions of the Frisch scheme. Therefore

$$\text{Maxcor}_{\mathcal{F}}(\boldsymbol{\Sigma}) = \max_{\hat{\boldsymbol{\Sigma}} \in \mathcal{D}} \left\{ \dim \left(\ker(\hat{\boldsymbol{\Sigma}}) \right) \right\}$$

being $\mathcal{D}$ the set of all the diagonal matrices satisfying (5), i.e., all the solutions of the Frisch scheme.

Theorem 1. *[6] If* $\text{Maxcor}_{\mathcal{F}}(\boldsymbol{\Sigma}) = 1$ *the following hold:*

1. *All the linear relations compatible with the Frisch scheme lie (by normalizing one entry to* 1*) inside a simplex of* $\mathbb{R}^{n-1}$ *whose vertices are the* n *OLS solutions.*
2. *There exists a one-to-one relation to between the points of the simplex and the solutions of the Frisch scheme.*

3 Robot parameters identification

As it is well known, the Euler-Lagrangian dynamic model of a n dof robot manipulator can be expressed in a linear form with respect to a vector of dynamic parameters $\boldsymbol{\theta}$ as

$$\boldsymbol{\Gamma}\left(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\right) \boldsymbol{\theta} = \boldsymbol{\tau} \tag{8}$$

where matrix $\boldsymbol{\Gamma}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) \in \mathbb{R}^{n \times 13n}$ depends only on the kinematic parameters of the robot and the joint position, velocity and acceleration $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}} \in \mathbb{R}^n$ respectively, [1]. Vector $\boldsymbol{\tau} \in \mathbb{R}^n$ collects the generalized forces applied to the joints and

$$\boldsymbol{\theta} = \left[\boldsymbol{\theta}_1^\top \ \boldsymbol{\theta}_2^\top \ \dots \ \boldsymbol{\theta}_n^\top \right]^\top \in \mathbb{R}^{13n}, \qquad \boldsymbol{\theta}_i \in \mathbb{R}^{13}$$

$\boldsymbol{\theta}_i$ is the vector collecting the parameters of link i, including:

- the mass of the link m_i;
- the first order moments: $m_i r_i^x$, $m_i r_i^y$, $m_i r_i^z$;
- the independent elements of the symmetric inertia matrix; expressed with respect to the origin of frame i: I_i^x, I_i^y, I_i^z, I_i^{xy}, I_i^{xz}, I_i^{yz};
- the actuator inertia I_i^m;
- the Coulomb and viscous friction coefficients μ_i and β_i respectively.

Usually, three conceptual steps are considered for parameter estimation: i) *model reduction*; ii) *planning of exciting trajectories*; iii) *actual parameter estimation.* In literature, many contributions have been proposed for improving the first two aspects, see e.g. [1,9,10,11,12]. In the next sections, a systematic procedure dealing with these two steps is presented, partially inspired by [1]; moreover, the novel identification procedure is illustrated.

3.1 Model Reduction

The first step is to define a reduced dynamic model of the robot starting from the full symbolic model (8). In robotics, it is well known that only restricted relative motion between links is allowed. Consequently it is impossible to estimate all the $13n$ parameters in $\boldsymbol{\theta}$. Indeed, the elements of $\boldsymbol{\theta}$ can be divided into three categories:

- $\boldsymbol{\theta}_a \in \mathbb{R}^{p_a}$, the *absolutely identifiable parameters*, i.e. parameters corresponding to structurally independent columns of $\boldsymbol{\Gamma}$;
- $\boldsymbol{\theta}_l \in \mathbb{R}^{p_l}$, the *parameters identifiable in linear combination*, i.e. parameters giving contribution to the joint torques only in linear combination with each other;
- $\boldsymbol{\theta}_n \in \mathbb{R}^{p_n}$, the *non-identifiable parameters*, i.e. parameters that do not give any contribution to the joint torques, corresponding to structurally null columns of $\boldsymbol{\Gamma}$.

Defining $c = 13n$, it yields $c = p_a + p_n + p_l$. The goal of the model reduction procedure is to find a vector of base parameters $\boldsymbol{\theta}_b \in \mathbb{R}^p$ containing the symbolic expression of the parameters characterizing the reduced model. In particular $\boldsymbol{\theta}_b = [\boldsymbol{\theta}_a^\top, \boldsymbol{\theta}_d^\top]^\top$ where $\boldsymbol{\theta}_d \in^{p_d}$ contains the sought linear combinations of the elements of $\boldsymbol{\theta}_l$. Obviously, it results $p_d < p_l$ and $p = p_a + p_d < c$. Therefore, matrix $\boldsymbol{\Gamma}$ can be replaced by the corresponding $n \times p$ reduced matrix $\boldsymbol{\Gamma}_b$ to obtain

$$\boldsymbol{\Gamma}_b(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})\, \boldsymbol{\theta}_b = \boldsymbol{\tau} \tag{9}$$

Since the purpose of this work is to define a systematic procedure which may be applied to any robot, a numerical approach based on *Singular Value Decomposition* (SVD) is adopted for model reduction, inspired by [10] [1]. Assume to have $m \geq 13$ observations of $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}$ abd $\boldsymbol{\tau}$ obtained by sampling a properly exciting trajectory (see 3.2). Let $r = mn$ and let $\boldsymbol{\Pi}_m \in \mathbb{R}^{r \times c}$ be the *regression matrix* obtained by computing $\boldsymbol{\Gamma}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$ in the m given samples[5].

$$\boldsymbol{\tau}_m \triangleq \begin{bmatrix} \boldsymbol{\tau}(t_1) \\ \boldsymbol{\tau}(t_2) \\ \vdots \\ \boldsymbol{\tau}(t_m) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Gamma}(\mathbf{q}(t_1), \dot{\mathbf{q}}(t_1), \ddot{\mathbf{q}}(t_1)) \\ \boldsymbol{\Gamma}(\mathbf{q}(t_2), \dot{\mathbf{q}}(t_2), \ddot{\mathbf{q}}(t_2)) \\ \vdots \\ \boldsymbol{\Gamma}(\mathbf{q}(t_m), \dot{\mathbf{q}}(t_m), \ddot{\mathbf{q}}(t_m)) \end{bmatrix} \boldsymbol{\theta} = \boldsymbol{\Pi}_m \boldsymbol{\theta} \tag{10}$$

with

$$\operatorname{rank}(\boldsymbol{\Pi}_m) = p < c$$

Since $\boldsymbol{\Pi}_m$ is never full rank, this appears to be a rank deficiency problem. As suggested by Gautier [10], the procedure to obtain the reduced model consists in two steps:

1. Find the rank of matrix $\boldsymbol{\Pi}_m$, that gives the number p of base parameters;

[5] In [1,10], $\boldsymbol{\Pi}_m$ is computed using m random samples, here we use the optimization procedure described in the next subsection to evaluate it.

2. Choose the base parameters from the standard ones by eliminating some of them which are regrouped to others in linear combinations (the regrouping relations will be determined in this step).

In the following a systematic procedure to accomplish this task is presented.

The parameters $\boldsymbol{\theta_a}$ and $\boldsymbol{\theta_n}$
By performing a SVD of $\mathbf{\Pi}_m$, the following factorization is obtained:

$$\mathbf{\Pi}_m = \mathbf{U}\mathbf{S}\mathbf{V}^\top = \mathbf{U}\begin{bmatrix}\mathbf{S}_{11} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{V}_1^\top\\ \mathbf{V}_2^\top\end{bmatrix} \tag{11}$$

where $\mathbf{U}\in\mathbb{R}^{r\times r}$, $\mathbf{S}\in\mathbb{R}^{r\times c}$, $\mathbf{V}_1\in\mathbb{R}^{c\times p}$, $\mathbf{V}_2\in\mathbb{R}^{c\times(c-p)}$, $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices and $\mathbf{S}_{11}\in\mathbb{R}^{p\times p}$ is a diagonal matrix containing (in non increasing order) the *non null* singular values of $\mathbf{\Pi}_m$. By substituting (11) in (10) one obtains

$$\boldsymbol{\tau}_m = \mathbf{U}\begin{bmatrix}\mathbf{S}_{11}\mathbf{V}_1^\top\boldsymbol{\theta}\\ 0\end{bmatrix}$$

The following considerations can be made, [1,10]:

- the *non-identifiable* parameters $\boldsymbol{\theta}_n$ correspond to the null columns of $\mathbf{V}_1^\top$
- the *absolutely identifiable* parameters $\boldsymbol{\theta}_a$ correspond to the null columns of $\mathbf{V}_2^\top$
- the remaining parameters are identifiable only in linear combinations.

Thus, we need to know these linear combinations to define a set of base parameters.

The $\boldsymbol{\theta_d}$ parameters
Consider matrix $\bar{\mathbf{\Pi}}_m\in\mathbb{R}^{r\times p_l}$ obtained eliminating from $\mathbf{\Pi}_m$ all the columns corresponding to the non-identifiable and absolutely identifiable parameters. A second SVD may be performed on $\bar{\mathbf{\Pi}}_m$:

$$\bar{\mathbf{\Pi}}_m = \bar{\mathbf{U}}\bar{\mathbf{S}}\bar{\mathbf{V}}^\top = \bar{\mathbf{U}}\begin{bmatrix}\bar{\mathbf{S}}_{11} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\bar{\mathbf{V}}_1^\top\\ \bar{\mathbf{V}}_2^\top\end{bmatrix}$$

with

$$\bar{\mathbf{S}}_{11}\in\mathbb{R}^{p_d\times p_d}\quad \bar{\mathbf{V}}_1\in\mathbb{R}^{p_l\times p_d}\quad \bar{\mathbf{V}}_2\in\mathbb{R}^{p_l\times(p_l-p_d)}$$

where $p_d = \mathrm{rank}(\bar{\mathbf{\Pi}}_m) = (p-p_a)$. It holds

$$\bar{\mathbf{\Pi}}_m\bar{\mathbf{V}}_2 = 0 \tag{12}$$

and thus the columns of $\bar{\mathbf{V}}_2$ define the linear combinations among the columns of $\bar{\mathbf{\Pi}}_m$. From (12), it follows

$$\bar{\mathbf{\Pi}}_m\boldsymbol{\theta}_l = \bar{\mathbf{\Pi}}_m(\boldsymbol{\theta}_l + \bar{\mathbf{V}}_2\boldsymbol{\theta}^\star) \tag{13}$$

where $\boldsymbol{\theta}^\star \in \mathbb{R}^{(p_l - p_d)}$ is an *arbitrary* vector. Hence, there exists an infinite number of vectors

$$\boldsymbol{\theta}_r = \boldsymbol{\theta}_l + \bar{\mathbf{V}}_2 \boldsymbol{\theta}^\star \tag{14}$$

which satisfy (13). The goal of this procedure is to linearly combine the components of $\boldsymbol{\theta}_l$ in p_d linearly independent elements to end up with a parameter vector $\boldsymbol{\theta}_d$ which completes the base parameter vector. We report the following result which provides a closed form solution for $\boldsymbol{\theta}_d$ [10].

Theorem 2. *There exists a permutation matrix* $\mathbf{P}$ *such that*

$$\mathbf{P}^\top \bar{\mathbf{V}}_2 = \begin{bmatrix} \bar{\mathbf{V}}_{21} \\ \bar{\mathbf{V}}_{22} \end{bmatrix}$$

with $\bar{\mathbf{V}}_{22}$ *invertible. Then, being* $\mathbf{P}^\top \boldsymbol{\theta}_l = \begin{bmatrix} \boldsymbol{\theta}_{l1} & \boldsymbol{\theta}_{l2} \end{bmatrix}^\top$ *, and* $\mathbf{P}^\top \boldsymbol{\theta}_r = \begin{bmatrix} \boldsymbol{\theta}_d & \boldsymbol{\theta}_{d\star} \end{bmatrix}^\top$ *, it holds that a solution of (14) with* $\boldsymbol{\theta}_{d\star} = 0$ *is*

$$\boldsymbol{\theta}_d = \boldsymbol{\theta}_{l1} - \bar{\mathbf{V}}_{21} \bar{\mathbf{V}}_{22}^{-1} \boldsymbol{\theta}_{l2}$$

To end up with the reduced model (9) the matrix $\boldsymbol{\Gamma}_b$ has to be computed starting from $\boldsymbol{\Gamma}$. In particular

$$\boldsymbol{\Gamma}_b = \begin{bmatrix} \boldsymbol{\Gamma}_a & \boldsymbol{\Gamma}_d \end{bmatrix} \in \mathbb{R}^{n \times p} \tag{15}$$

where $\boldsymbol{\Gamma}_a \in \mathbb{R}^{n \times p_a}$ is the matrix whose columns correspond to the parameters $\boldsymbol{\theta}_a$ and $\boldsymbol{\Gamma}_d \in \mathbb{R}^{n \times p_d}$ is the matrix whose columns correspond to the parameters $\boldsymbol{\theta}_d$. Now, a novel result useful to compute Y_d in closed form is presented.

Theorem 3. *Let* $\boldsymbol{\Gamma}_{l1} \in \mathbb{R}^{n \times p_d}$ *and* $\boldsymbol{\Gamma}_{l2} \in \mathbb{R}^{n \times (p_l - p_d)}$ *be the matrices corresponding to the elements of* $\boldsymbol{\theta}_{l1}$ *and* $\boldsymbol{\theta}_{l2}$ *respectively. It results*

$$\boldsymbol{\Gamma}_d = (\boldsymbol{\Gamma}_{l1}^\top - \bar{\mathbf{V}}_{21} \bar{\mathbf{V}}_{22}^{-1} \boldsymbol{\Gamma}_{l2}^\top)^\top \tag{16}$$

Proof. $\boldsymbol{\Gamma}_d$ must be constructed combining the columns of $\boldsymbol{\Gamma}_{l1}$ and $\boldsymbol{\Gamma}_{l2}$ with the same rules with which the elements of $\boldsymbol{\theta}_{l1}$ and $\boldsymbol{\theta}_{l2}$ in $\boldsymbol{\theta}_d$ are combined. Naming $\boldsymbol{\Gamma}_{d,i} \in \mathbb{R}^d$ the i-th row of $\boldsymbol{\Gamma}_d$ and, similarly, $\boldsymbol{\Gamma}_{l1,i} \in \mathbb{R}^{p_d}$ and $\boldsymbol{\Gamma}_{l2,i} \in \mathbb{R}^{(p_l - p_d)}$ the i-th rows of $\boldsymbol{\Gamma}_{l1}$ and $\boldsymbol{\Gamma}_{l2}$, we have

$$\boldsymbol{\Gamma}_{d,i}^\top = \boldsymbol{\Gamma}_{l1,i}^\top - \bar{\mathbf{V}}_{21} \bar{\mathbf{V}}_{22}^{-1} \boldsymbol{\Gamma}_{l2,i}^\top$$

Since this is true for each row of $\boldsymbol{\Gamma}_d$ we can write

$$\boldsymbol{\Gamma}_d^\top = \boldsymbol{\Gamma}_{l1}^\top - \bar{\mathbf{V}}_{21} \bar{\mathbf{V}}_{22}^{-1} \boldsymbol{\Gamma}_{l2}^\top \Leftrightarrow \boldsymbol{\Gamma}_d = (\boldsymbol{\Gamma}_{l1}^\top - \bar{\mathbf{V}}_{21} \bar{\mathbf{V}}_{22}^{-1} \boldsymbol{\Gamma}_{l2}^\top)^\top$$

■

Remark 1. $\bar{\mathbf{V}}_{22}$ is function of the permutation matrix $\mathbf{P}$. We know that $\bar{\mathbf{V}}_{22} \in \mathbb{R}^{(p_l - p_d) \times (p_l - p_d)}$. The number $n_{\mathbf{P}}$ of existing different permutation matrices $\mathbf{P}$ is

equal to the number of possible combinations (without repetitions) of the p_l rows of $\bar{\mathbf{V}}_2$ in sets of $(p_l - p_d)$ elements[6]

$$n_{\mathbf{P}} \triangleq C_{p_l,(p_l-p_d)} = \binom{p_l}{p_l - p_d} = \frac{p_l!}{p_d!(p_l - p_d)!}$$

Therefore we need to set up an optimization problem to choose a proper permutation matrix. In this work, in order to guarantee the regularity of $\bar{\mathbf{V}}_{22}$, $\mathbf{P}$ has been selected solving the following optimization problem:

$$\mathbf{P} = \underset{\mathbf{P}\in\mathcal{P}}{\arg\min} \left(\alpha_1 \operatorname{cond}(\bar{\mathbf{V}}_{22}) + \alpha_2 \frac{1}{\det(\bar{\mathbf{V}}_{22})} \right) \tag{17}$$

where $\operatorname{cond}(\bar{\mathbf{V}}_{22})$ is the condition number of $\bar{\mathbf{V}}_{22}$, α_1 and α_2 are arbitrary coefficients and $\mathcal{P}$ is the set of all permutation matrices of appropriate dimension.

3.2 Exciting Trajectory Planning

In this subsection, a systematic way to plan an optimal exciting trajectory to be sampled in m time instances is presented.

The sensitivity of linear estimations to noise and errors is associated with the conditioning number of the regression matrix $\mathbf{\Pi}_m$ (see e.g. [1,3,11,12,13]), which should be small when evaluated along a suitable exciting trajectory. The latter should also not excite any unmodeled dynamic effects such as joint elasticity or link flexibility that would naturally lead to unreliable estimates [13].
An exciting trajectory is defined by a sequence of m triplets

$$\mathbf{z}_i \triangleq \left(\mathbf{q}(t_i), \dot{\mathbf{q}}(t_i), \ddot{\mathbf{q}}(t_i)\right) \qquad i = 1, \ldots, m \tag{18}$$

which give values to $\mathbf{\Gamma}$ along the trajectory, then used to build $\mathbf{\Pi}_m$:

$$\mathbf{\Pi}_m = \begin{bmatrix} \mathbf{\Gamma}(\mathbf{z}_1) \\ \mathbf{\Gamma}(\mathbf{z}_2) \\ \vdots \\ \mathbf{\Gamma}(\mathbf{z}_m) \end{bmatrix}$$

The m points[7] of the optimal exciting trajectory are the solutions of the nonlinear constrained optimization problem formulated as follows.

Let $\mathbf{z} \triangleq (\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m)$, $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_m)$ with $\mathbf{n}_i \triangleq \left(\mathbf{n}_{\mathbf{q},i}, \mathbf{n}_{\dot{\mathbf{q}},i}, \mathbf{n}_{\ddot{\mathbf{q}},i}\right)$ being made up of identically independently distributed samples of multivariate zero–mean normal distributions with unitary variance, i.e. $\mathbf{n}_{\mathbf{q},i}, \mathbf{n}_{\dot{\mathbf{q}},i}, \mathbf{n}_{\ddot{\mathbf{q}},i} \sim \mathcal{N}(\mathbb{0}_n, \mathbf{I}_n)$. Furthermore, let

$$\mathbf{\Psi}(\mathbf{z}, \mathbf{n}) \triangleq \frac{1}{nm} \left[\mathbf{\Pi}_m(\mathbf{z}+\mathbf{n}) - \mathbf{\Pi}_m(\mathbf{z})\right]^\top \left[\mathbf{\Pi}_m(\mathbf{z}+\mathbf{n}) - \mathbf{\Pi}_m(\mathbf{z})\right]$$

[6] In [1] it is erroneously stated that $n_{\mathbf{P}} = \infty$. Although this is not theoretically correct, in robotics $n_{\mathbf{P}}$ is a huge number (for the PUMA 560 $n_{\mathbf{P}} \sim 5 \cdot 10^8$)

[7] In order not to result in an underdetermined system, $nm > p$ must yield

and

$$\psi(\mathbf{z},\mathbf{n}) \triangleq \left\| \boldsymbol{\Psi}(\mathbf{z},\mathbf{n}) - \mathrm{diag}\left(\mathrm{diag}\left(\boldsymbol{\Psi}(\mathbf{z},\mathbf{n})\right)\right) \right\|_{\infty}$$

. The optimization problem is

$$\begin{aligned} \underset{\mathbf{z}}{\text{minimize}} \quad & \lambda_1 \mathrm{cond}\left(\boldsymbol{\Pi}_m(\mathbf{z})\right) + \lambda_2 \frac{1}{\sigma_{min}\left(\boldsymbol{\Pi}_m(\mathbf{z})\right)} + \psi(\mathbf{z},\mathbf{n}) \\ \text{subject to} \quad & \forall i \quad \mathbf{q}(t_i) \in \mathcal{W}_{\mathbf{q}} \\ & \forall i \quad \dot{\mathbf{q}}(t_i) \in \mathcal{W}_{\dot{\mathbf{q}}} \\ & \forall i \quad \ddot{\mathbf{q}}(t_i) \in \mathcal{W}_{\ddot{\mathbf{q}}}. \end{aligned} \tag{19}$$

where $\sigma_{min}(\boldsymbol{\Pi}_m)$ is the minimum singular value of $\boldsymbol{\Pi}_m$, and λ_1, λ_2 two arbitrary coefficients. $\mathcal{W}_{\mathbf{q}}$, $\mathcal{W}_{\dot{\mathbf{q}}}$ and $\mathcal{W}_{\ddot{\mathbf{q}}}$ are the sets of admissible joint positions, velocities and accelerations, respectively. The constraints have been set to take into account the physical limits of the manipulator.

Furthermore, the component $\psi(\mathbf{z},\mathbf{n})$ of the cost function is used to increase the performance of the Frisch scheme estimation. In fact, $\boldsymbol{\Psi}(\mathbf{z},\mathbf{n})$ can be seen as the noise sample covariance matrix corresponding to the synthetic addite noise $\mathbf{n}$. Thus, by minimizing $\psi(\mathbf{z},\mathbf{n})$, we obtain an exciting trajectory along which the off–diagonal terms of the noise sample covariance matrix are suppressed, enforcing the assumptions of the Frisch scheme. Besides, due to the nonlinear terms of $\mathbf{q}$, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$ in the entries of $\boldsymbol{\Pi}_m$, it is very difficult to obtain a trajectory in which independent additive noise on the $\mathbf{q}$, $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ leads to a diagonal noise covariance matrix.

Note that [1] only considers the first part of the cost function (without $\psi(\mathbf{z},\mathbf{n})$) as the are interested in performing a least squares estimation.

The number of *decision variables* of this optimization problem is $3nn$, being each measurement point $\mathbf{z}$ defined by $3n$ variables. Hence, this number varies linearly with m and it is thus necessary to determine a trade off between the possible advantages derived by considering high values of m and the efficiency of the optimization algorithm.

Once the optimization problem has been solved, the m optimal measurement points must be interpolated in a smooth way in order to impose continuity of accelerations; this can be achieved e.g. with 3^{rd} order splines or 5^{th} order polynomial trajectories, [14].

3.3 Identification procedure

Assume that a proper reduced model (9) has been derived. In order to estimate the parameters $\boldsymbol{\theta}_b$, we impose to the robot a motion defined by a properly chosen exciting trajectory and we record the sensor measurements of $\mathbf{q}(t)$, $\dot{\mathbf{q}}(t)$, $\ddot{\mathbf{q}}(t)$ and $\boldsymbol{\tau}(t)$ along this trajectory. Matrix $\boldsymbol{\Gamma}_b$ is computed and the joint torques are sampled in the m time instants $t_1, t_2, \ldots, t_m$ corresponding to the optimized measurement points. They are then stacked in a matrix $\boldsymbol{\Pi}_{m,b}$ and in a vector $\boldsymbol{\tau}_m$

as follows

$$\boldsymbol{\Pi}_{m,b} = \begin{bmatrix} \boldsymbol{\Gamma}_b(\mathbf{q}(t_1), \dot{\mathbf{q}}(t_1), \ddot{\mathbf{q}}(t_1)) \\ \boldsymbol{\Gamma}_b(\mathbf{q}(t_2), \dot{\mathbf{q}}(t_2), \ddot{\mathbf{q}}(t_2)) \\ \vdots \\ \boldsymbol{\Gamma}_b(\mathbf{q}(t_m), \dot{\mathbf{q}}(t_m), \ddot{\mathbf{q}}(t_m)) \end{bmatrix} \quad \boldsymbol{\tau}_m = \begin{bmatrix} \boldsymbol{\tau}(t_1) \\ \boldsymbol{\tau}(t_2) \\ \vdots \\ \boldsymbol{\tau}(t_m) \end{bmatrix}$$

From (9), we can write

$$\boldsymbol{\Pi}_{m,b}\boldsymbol{\theta}_b = \boldsymbol{\tau}_m \tag{20}$$

from which

$$\overbrace{\left[\boldsymbol{\Pi}_{m,b}, -\boldsymbol{\tau}_m\right]}^{\boldsymbol{\Pi}_{m,b,e}} \underbrace{\begin{bmatrix} \boldsymbol{\theta}_b \\ 1 \end{bmatrix}}_{\boldsymbol{\theta}_{b,e}} = \mathbb{0}_n \tag{21}$$

Defining now the *covariance matrix* $\boldsymbol{\Sigma}_e$ as

$$\boldsymbol{\Sigma}_e \triangleq \frac{1}{m}\boldsymbol{\Pi}_{m,b,e}^{\top}\boldsymbol{\Pi}_{m,b,e} \in \mathbb{R}^{(p+1)\times(p+1)} \tag{22}$$

equation (21) can be rewritten (equivalently to (4)) as

$$\boldsymbol{\Sigma}_e\, \boldsymbol{\theta}_{b,e} = 0 \tag{23}$$

It is now possible to implement the Frisch estimation scheme:

1. First of all the simplex of solutions in the parameter space is computed by performing $p+1$ OLS solutions corresponding to the problem (23). Then, if we normalize a parameter to 1, each OLS solution becomes a vertex of the simplex.
2. The interior of the simplex represents the region in which any solution $\boldsymbol{\theta}_{b,e}$ can take values according to the Frisch scheme, and thus it represents a region where the "best" parameter vector $\boldsymbol{\theta}_{b,e}$ can be selected.

The following remarks highlight some interesting features of the application of this procedure in robotics.

Remark 2. The application of the Frisch scheme to (23) represents a pure black box identification approach. Nevertheless we have a very powerful knowledge of the vector $\boldsymbol{\theta}_{b,e}$: it stores the base dynamic parameters of the robot whose symbolic expression is known. Indeed this knowledge can be used to establish proper criteria to seek $\boldsymbol{\theta}_{b,e}$ inside feasible subregions of the simplex[8]. Furthermore notice that the last element of $\boldsymbol{\theta}_{b,e}$ must be equal to 1 by construction and thus the OLS solutions should be computed by accomplishing this normalization.

Remark 3. The assumption $\mathsf{Maxcor}_{\mathcal{F}}(\boldsymbol{\Sigma}_e) = 1$ is very reasonable since (23) is the model of a robot manipulator which has been properly reduced to a base set of parameters and conditioned by an exciting trajectory which should minimize the condition number of $\boldsymbol{\Sigma}_e$.

[8] e.g. masses and friction coefficients cannot be negative

3.4 Solution Selection and Validation

In order to select a unique solution from the simplex, a systematic procedure can be developed based on the *bounding–box recursive Frisch scheme* (BBRF) [15]. This estimation technique aims at reducing the size of the space of solutions of the identification problem in the context the Frisch scheme. In particular, the BBRF solution can be computed iteratively from the regression matrix $\mathbf{\Pi}_{m,b,e}$ as thoroughly described in [15]. It consists in lower and upper bounds $\mathbf{l}(k),\ \mathbf{u}(k) \in \mathbb{R}^p$ (k indicates the algorithm iteration),

$$\mathbf{l}(k) \triangleq [l_1(k),\dots,l_p(k)]^\top, \quad \mathbf{u}(k) \triangleq [u_1(k),\dots,u_p(k)]^\top$$

in which the true vector of parameters, i.e. the exact solution of the identification problem, is supposed to be laying according to the assumptions of the Frisch scheme.

Let $\boldsymbol{\theta}_{\min}(\mathbf{\Sigma}),\ \boldsymbol{\theta}_{\max}(\mathbf{\Sigma})$ be the vectors containing the minimum and maximum estimations of each parameter among the vertices of simplex associated to $\mathbf{\Sigma}$, i.e. they define the *axis aligned bounding–box* enclosing the simplex [9]. A pseudo–code of the BBRF in the context of robot identification is provided in Agorithm 1.

In this phase, the post–identification degree–of–freedom offered by the Frisch scheme become significant: in those cases in which it is known the sign of the parameter, it sufficient to set the lower or upper bound of that parameter to zero. E.g., suppose θ_i to be a friction coefficient, then we will set $l_i(0) = 0$. The physical consistency will be then ensured by the monotonicity property of $\mathbf{l}$ and $\mathbf{u}$ given by the algorithm, i.e.

$$\forall i,k \quad l_i(k+1) \geq l_i(k) \ \wedge \ u_i(k+1) \leq u_i(k)$$

[9] Note that, in the case of the robot model, the normalized entry of $\boldsymbol{\theta}_{b,e}$ is always the last one

Algorithm 1 Bounding Box Recursive Frisch Scheme

Inputs: $\mathbf{\Pi}_{m,b,e},\ w,$
Initialize: $\mathbf{X}_0 = [\mathbf{\Pi}_{m,b,e}]_{1:w+1,:}$
$\mathbf{\Sigma}_0 = \frac{1}{w}\mathbf{X}_0^\top \mathbf{X}_0$
$\mathbf{l}(0) = \boldsymbol{\theta}_{\min}(\mathbf{\Sigma}_0)$,
$\mathbf{u}(0) = \boldsymbol{\theta}_{\max}(\mathbf{\Sigma}_0)$
$\mathrm{k} = 2$
while $k \leq nm - w - 1$ **do**,
  $\mathbf{X}(k) = [\mathbf{\Pi}_{m,b,e}]_{i:w+i}$,
  $\mathbf{\Sigma}(k) = \frac{1}{w}\mathbf{X}(k)^\top \mathbf{X}(k)$,
  $l_i(k) = \max\{l_i(k-1), \theta_{\min,i}(\mathbf{\Sigma}(k))\} \quad \forall j = 1,\dots,p,$
  $u_i(k) = \min\{u_i(k-1), \theta_{\max,i}(\mathbf{\Sigma}(k))\} \quad \forall j = 1,\dots,p,$
  $k \leftarrow k+1,$
Outputs: $\mathbf{l}(k),\ \mathbf{u}(k)$

Link (i)	α_i [deg]	θ_i	a_i [m]	d_i [m]
1	0	q_1	1	0
2	0	q_2	1	0

Table 1. Denavit-Hartenberg parameters of a 2-DOF planar robot

Let k_{tot} the total number of BBRF iterations, a single consistent solution of the Frisch scheme can be computed, for example, by minimizing a regularized Euclidian norm cost function in the physically feasible region given by the Frisch scheme i.e.

$$\begin{aligned} &\underset{\boldsymbol{\theta}_b^*}{\text{minimize}} \quad \|\boldsymbol{\Pi}_{m,b}\boldsymbol{\theta}_b^* - \boldsymbol{\tau}_m\|_2 + \|\boldsymbol{\theta}_b^*\|_2 \\ &\text{subject to} \quad \forall i \quad l_i(k_{tot}) \le \theta_{b,i}^* \le u_i(k_{tot}) \end{aligned} \tag{24}$$

4 Case Study: 2-DOF manipulator

To validate the above procedure, an identification experiment on a planar 2-DOF robot has been performed via simulation. A routine in *Mathematica* © has been implemented, taking as input the Denavit-Hartemberg parameters of the manipulator (reported in Table 1) and computing symbolically the matrix $\boldsymbol{\Gamma} \in \mathbb{R}^{2\times 26}$ and the corresponding vector of parameters $\boldsymbol{\theta} \in \mathbb{R}^{26}$. These data are then exported in *Matlab* © where the rest of the steps are performed[10]. The procedure for trajectory planning described in 3.2 has been implemented with $m = 100$, $\lambda_1 = \lambda_2 = 1$.

Remark 4. It is worth noticing that, for the sake of model reduction, we compute the rank deficient regression matrix $\boldsymbol{\Pi}_m$ along an optimized trajectory $\mathbf{z}_1$ and not with random samples as in [1,10].

After the first SVD, (3.1), it results:

$$\text{rank}(\boldsymbol{\Pi}_m) = 9, \;\; p_n = 14, \;\; p_a = 5$$

Thus, four parameters are identifiable only in linear combinations ($p_d = 4$, $p_l = 7$). After the second SVD (3.1), we have:

$$\begin{cases} \bar{\boldsymbol{\Pi}}_m \in \mathbb{R}^{40\times 7}, \quad \bar{\mathbf{V}} \in \mathbb{R}^{7\times 7}, \\ \text{rank}(\bar{\boldsymbol{\Pi}}_m) = 4, \;\; \bar{\mathbf{V}}_2 \in \mathbb{R}^{7\times 3}, \;\; \bar{\mathbf{V}}_{22} \in \mathbb{R}^{3\times 3} \end{cases}$$

The number of possible permutation matrices $\mathbf{P}$ is

$$C_{32,15} = \frac{7!}{3!4!} = 35$$

We select the one which solves (17) with $\alpha_1 = \alpha_2 = 1$.

[10] The whole code to reproduce every step of this paper is available at https://github.com/massastrello/Robot-Frisch

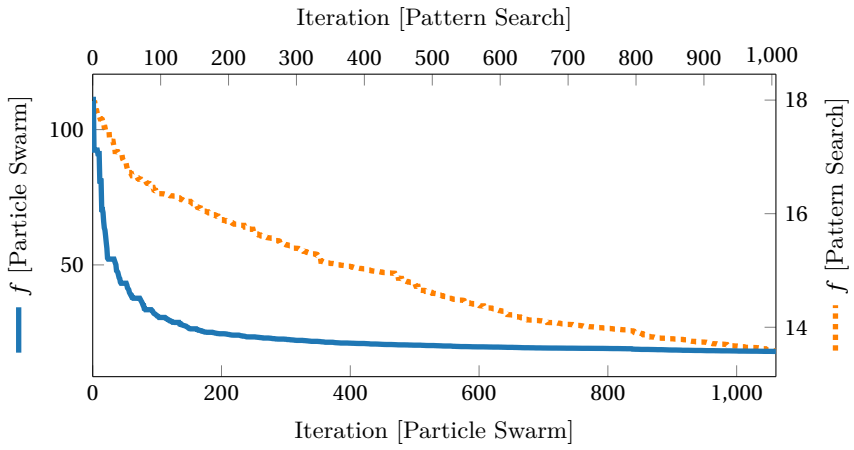

Fig. 1. Convergence of the optimization for $\mathbf{z}_2$

Remark 5. For different optimization trials of $\mathbf{z}_1$, the model reduction procedure leads always to the same result, validating the proposed numerical method. The result $\mathrm{rank}(\bar{\mathbf{\Pi}}_m) = 4$ confirms our expectations: it is equal to the number of linear combinations of parameters needed to complete the set of base parameters.

The symbolic expression of the reduced matrix $\mathbf{\Gamma}_b$ is then derived as in (15) and an exciting trajectory $\mathbf{z}_2$ is computed for the new model. The procedure to solve (19) is carried out with an hybrid optimization technique which combines a *Particle Swarm Optimization* and a *Pattern Search* algorithm [16,17]. The optimization results are reported in Table 2 while the descent of the cost function during the optimization is shown in Fig. 1. Interpolation has been performed using 5^{th} order polynomials with a time interval of $1s$ between the samples. Note that the regression matrix has been built using all the point of the interpolated trajectory measured with a sampling rate of 50Hz, rather then only the optimized ones. A total of 4951 data points have been obtained. The first $30s$ of the interpolated trajectory is showed in Fig. 2. Simulation of the data acquisition process has been implemented according to the following considerations:

– Joints positions, velocities, accelerations and torques are measurable variables. This values are comparable with commercial sensors;

Algorithm	Iterations	Function Evaluations	Cost func. val.
Particle Swarm	1061	106200	18.0100
Pattern Search	1006	293471	13.5739
Cost Function Value	$\mathrm{cond}(\mathbf{\Pi}_{m,b})$	σ_{max}	σ_{min}
13.5739	11.1564	74.5923	6.6860

Table 2. Optimization data of $\mathbf{z}_2$

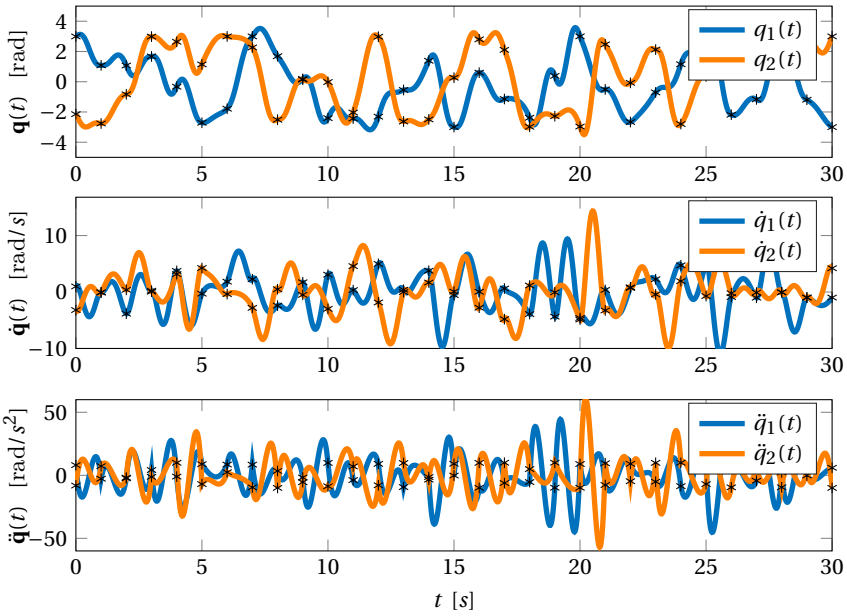

Fig. 2. First 30s of the interpolated trajectory (solid lines). The markers denote the optimized points of $\mathbf{z}_2$.

– Independent Gaussian noise $\mathbf{n}_q$, $\mathbf{n}_{\dot{q}}$ and $\mathbf{n}_{\ddot{q}}$ have been added to the joints positions, velocities and accelerations respectively with standard deviations $\sigma(\mathbf{n_q}) = 10^{-2}$, $\sigma(\mathbf{n_{\dot{q}}}) = \sigma(\mathbf{n_{\ddot{q}}}) = 2 \cdot 10^{-2}$;
– The reference reduced base parameters vector $\boldsymbol{\theta}_{b,\mathrm{ref}}$ has been defined according to the model reduction (see Table 3 for their symbolic expression). In particular the physical robot parameters have been chosen as: $\beta_1 = 0.1434$, $\beta_2 = 0.1391$, $\mu_1 = 0.3302$, $\mu_2 = 0.3576$, $m_1 = 21\mathrm{Kg}$, $m_2 = 10\mathrm{Kg}$, $I_1^z = 0.4667\mathrm{Kg}\cdot\mathrm{m}^2$, $I_2^z = 0.2333\mathrm{Kg}\cdot\mathrm{m}^2$, $I_2^m = 21.18\mathrm{Kg}\cdot\mathrm{m}^2$, $I_1^m = 12.10\mathrm{Kg}\cdot\mathrm{m}^2$, $r_1^x = 0.25\mathrm{m}$, $r_2^x = 0.25\mathrm{m}$. Note that these values have been partially inspired by [18].
– The nominal joint torque samples are derived from (20) using the reference parameters values: $\boldsymbol{\tau}_m = \boldsymbol{\Pi}_{m,b}\boldsymbol{\theta}_{b,\mathrm{ref}}$;
– Independent Gaussian noise $\mathbf{n}_{\boldsymbol{\tau}}$ is also added to each torque sample, with standard deviation $\sigma(\mathbf{n}_{\boldsymbol{\tau}}) = 1$;
– $\boldsymbol{\Pi}_{m,b,e} \in \mathbb{R}^{9902\times 10}$ is obtained from (21) and $\boldsymbol{\Sigma}_e \in \mathbb{R}^{10\times 10}$ as in (22).

Therefore the simplex of solutions in the parameters space of the Frisch scheme is determined by the OLS solutions associated to $\boldsymbol{\Sigma}_e$.

4.1 Results and comments

For the sake of reliability and repeatability of the simulation experiments, a Monte Carlo simulation of 250 runs has been performed re–sampling the noise

Parameter (θ_i)	$\boldsymbol{\theta}_{b,\text{ref}}$	$\boldsymbol{\theta}_{b,\min}(\boldsymbol{\Sigma}_e)$	$\boldsymbol{\theta}_{b,\max}(\boldsymbol{\Sigma}_e)$
β_1	0.1434	-3.4309 ± 0.4304	3.1086 ± 0.1459
β_2	0.1391	-2.7838 ± 0.3448	2.5173 ± 0.0955
μ_1	0.3302	-7.8714 ± 2.0704	17.4337 ± 1.2847
μ_2	0.3576	-7.0842 ± 1.7640	16.1686 ± 0.9681
I_2^m	21.1800	21.1735 ± 0.0011	21.1862 ± 0.0007
$m_1/2 + m_2/2 + m_1 r_1^x$	20.7500	20.7150 ± 0.0038	20.7717 ± 0.0014
$I_1^z + I_1^m/100 - m_1/4 - m/4$	-7.1623	-7.1677 ± 0.0010	-7.1548 ± 0.0006
$m_2/2 + m_2 r_2^x$	7.5000	7.4941 ± 0.0010	7.5061 ± 0.0008
$I_2^z - m_2/4$	-2.2667	-2.2883 ± 0.0036	-2.2398 ± 0.0019

Table 3. Results of the Frisch Scheme estimation.

each time. The results have then been averaged. The resulting simplex has 10 vertices and belongs to $\mathbb{R}^9$. Table 3 reports, for each base parameter, the reference value, the minimum and maximum estimation among the vertices (OLS solutions), i.e. the bounds of the bounding–box enclosing the simplex. Although each reference parameter is between the corresponding minimum and maximum values, the simplex appears to be wide spread (of some orders of magnitude) in the parameter space across the dimensions corresponding to the "smallest" parameters. Thus, the selection procedure is necessary to: first, consistently shrink the search space of the parameters, i.e. the estimation uncertainty, in a physically feasible manner and, second, select a single solution which can be used, as validation, to compute the reconstructed torques. The selelection procedure based in the BBRF has been implemented with $w = 10^3$ and $\boldsymbol{\theta}_b^*$ has been obtained solving (24) with the *Matlab* function *fmincon*. The absolute value of the estimation error $\mathbf{e}$ has been computed as

$$\mathbf{e} \triangleq [e_1, \dots, e_p]^\top : \quad e_i = |\theta_{b,i}^* - \theta_{b,\text{ref},i}|.$$

$\boldsymbol{\theta}_{b,\text{ref}}$	$\boldsymbol{\theta}_b^*$	$\mathbf{e}$
0.1434	0.1427 ± 0.0084	0.0061 ± 0.0057
0.1391	0.1397 ± 0.0052	0.0041 ± 0.0032
0.3302	0.3281 ± 0.0318	0.0209 ± 0.0240
0.3302	0.3563 ± 0.0221	0.0176 ± 0.0134
21.18	21.1810 ± 0.0014	0.0013 ± 0.0011
20.75	20.7489 ± 0.0020	0.0018 ± 0.0015
-7.1623	-7.1620 ± 0.0005	0.0005 ± 0.0004
7.5	7.5006 ± 0.0015	0.0013 ± 0.0010
-2.2667	-2.2691 ± 0.0028	0.0029 ± 0.0022

Table 4. Results of the BBRF–based solution selection process.

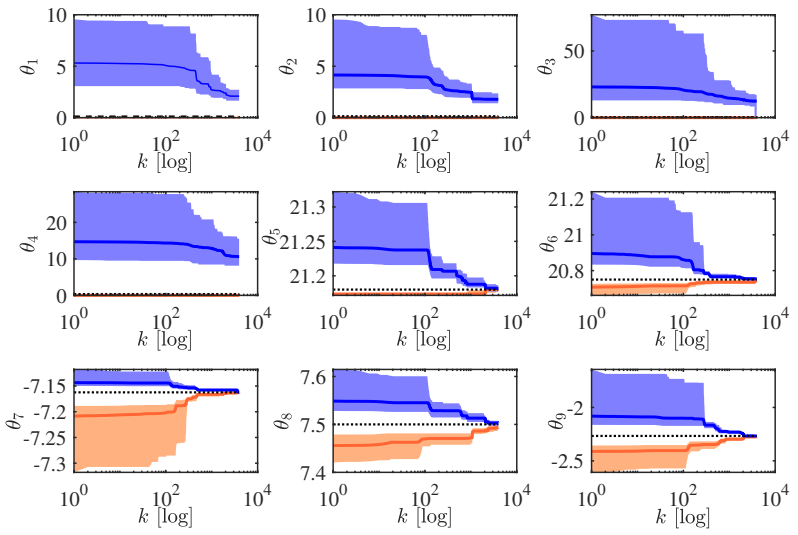

Fig. 3. Average among the Monte Carlo runs of the lower (solid orange) and upper bounds (solid blue) of the BBRF algorithm. The dotted black line indicates the real value of the parameter. The coloured shadows are the minimum and maximum values of the bounds among the Monte Carlo runs.

The optimized selected parameter and the estimation error are reported in Table 4. The evolution of the bounds $\mathbf{l}(k)$ and $\mathbf{u}(k)$ are shown in Fig. 3. It can be noticed how the bounds shrinks monotonically around the true parameter maintaining in between the true value while preserving the physical consistency. Note that the lower bounds of the first four base parameters, i.e. the friction coefficients β_1, β_2, μ_1 and μ_2, has been initialized to zero having the prior on their positiveness, as remarked in 3.4. As a final validation step, the torques where recomputed with the selected parameter vector $\boldsymbol{\theta}_b^*$. The reconstructed torques are compared with the measured ones in Fig. 4.

5 Conclusions

In this paper, a novel identification procedure for robot manipulators has been proposed. Firstly, systematic methods for model reduction and for planning exciting trajectories have been presented. Then, the Frisch scheme has been used to compute a simplex of possible parameters estimate, generalizing previously known results. As a matter of fact, with standard techniques it is possible to obtain unrealistic results (e.g. negative masses and friction coefficients) and in any case the basic assumption in OLS approaches (in which only one variable is affected by noise) does not appear to be physically reasonable.

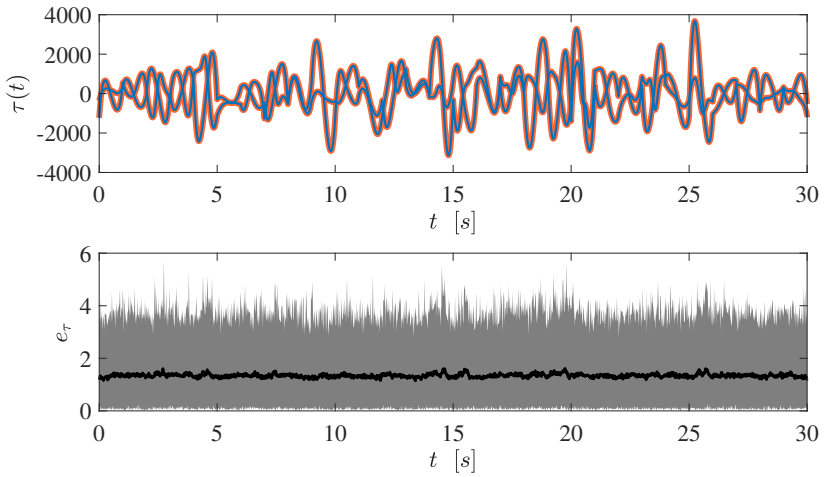

Fig. 4. [Above] Reconstructed torques (blue) versus measured torques (orange) (Average among Monte Carlo runs). [Below] Absolute value torque reconstruction error. The shadow indicate the minimum and maximum value of the torque error among the Monte Carlo runs. For clarity of the figure, only the first $30s$ of data have been reported

The values of parameters computed by the Frisch scheme are not unique, and lie in a simplex whose vertices are easy to be computed. The knowledge of the symbolic expression of the parameters can be used to choose the most suitable (and feasible) values, e.g. using the BBRF technique.

Future research aims at improving optimal criteria to select a single parameter value form the simplex. For this purpose, several methods can be adopted, involving the theory of independent experiments in the Frisch scheme framework [19], together with feasible region selection. Comparisons with other state–of–the–art approaches will also be made.

References

1. Gianluca Antonelli, Fabrizio Caccavale, and Pasquale Chiacchio. A systematic procedure for the identification of dynamic parameters of robot manipulators. *Robotica*, 17(4):427–435, 1999.
2. Christopher G Atkeson, Chae H An, and John M Hollerbach. Estimation of inertial parameters of manipulator loads and links. *The International Journal of Robotics Research*, 5(3):101–119, 1986.
3. G Calafiore, Marina Indri, and Basilio Bona. Robot dynamic calibration: Optimal excitation trajectories and experimental parameter estimation. *Journal of robotic systems*, 18(2):55–68, 2001.
4. M. Gautier and W. Khalil. On the identification of the inertial parameters of robots. In *Proceedings of the 27th IEEE Conference on Decision and Control*, pages 2264–2269 vol.3, Dec 1988.

5. Cristóvão Jorge Silva Duarte Sousa. *Dynamic model identification of robot manipulators: Solving the physical feasibility problem*. PhD thesis, 2015.
6. Guidorzi, Roberto and Diversi, Roberto and Soverini, Umberto. The Frisch scheme in algebraic and dynamic identification problems. *Kybernetika*, 44(5):585–616, 2008.
7. Rudolf E Kalman. Identification from real data. In *Current developments in the interface: Economics, Econometrics, Mathematics*, pages 161–196. Springer, 1982.
8. Roberto Guidorzi. *Multivariable system identification: from observations to models*. Bononia University Press, 2003.
9. Maxime Gautier and Wisama Khalil. A direct determination of minimum inertial parameters of robots. In *Robotics and Automation, 1988. Proceedings., 1988 IEEE International Conference on*, pages 1682–1687. IEEE, 1988.
10. Maxime Gautier. Numerical calculation of the base inertial parameters of robots. *Journal of Field Robotics*, 8(4):485–506, 1991.
11. Maxime Gautier and Wisama Khalil. Exciting trajectories for the identification of base inertial parameters of robots. *The International journal of robotics research*, 11(4):362–375, 1992.
12. Jan Swevers, Chris Ganseman, D Bilgin Tukel, Joris De Schutter, and Hendrik Van Brussel. Optimal robot excitation and identification. *IEEE transactions on robotics and automation*, 13(5):730–740, 1997.
13. Bruno Siciliano, Lorenzo Sciavicco, Luigi Villani, and Giuseppe Oriolo. *Robotics: modelling, planning and control*. Springer Science & Business Media, 2010.
14. Luigi Biagiotti and Claudio Melchiorri. *Trajectory planning for automatic machines and robots*. Springer Science & Business Media, 2008.
15. S. Massaroli, R. Miyagusuku, F. Califano, A. Faragasso, A. Yamashita, and H. Asama. A novel recursive linear estimator based on the frisch scheme. In *2019 12th Asian Control Conference (ASCC)*, pages 1101–1106, June 2019.
16. James Kennedy. Particle swarm optimization. In *Encyclopedia of machine learning*, pages 760–766. Springer, 2011.
17. Yuhui Shi and Russell Eberhart. A modified particle swarm optimizer. In *Evolutionary Computation Proceedings, 1998. IEEE World Congress on Computational Intelligence., The 1998 IEEE International Conference on*, pages 69–73. IEEE, 1998.
18. Loredana Zollo, Edoardo Lopez, Luca Spedaliere, Nicolas Garcia Aracil, and Eugenio Guglielmelli. Identification of dynamic parameters for robots with elastic joints. *Advances in Mechanical Engineering*, 7(2):843186, 2015.
19. Stefano Massaroli, Renato Myagusuku, Federico Califano, Claudio Melchiorri, Atsushi Yamashita, and Hajime Asama. Recursive algebraic frisch scheme: a particle-based approach. *IFAC-PapersOnLine*, 51(25):409 – 415, 2018. 9th IFAC Symposium on Robust Control Design ROCOND 2018.